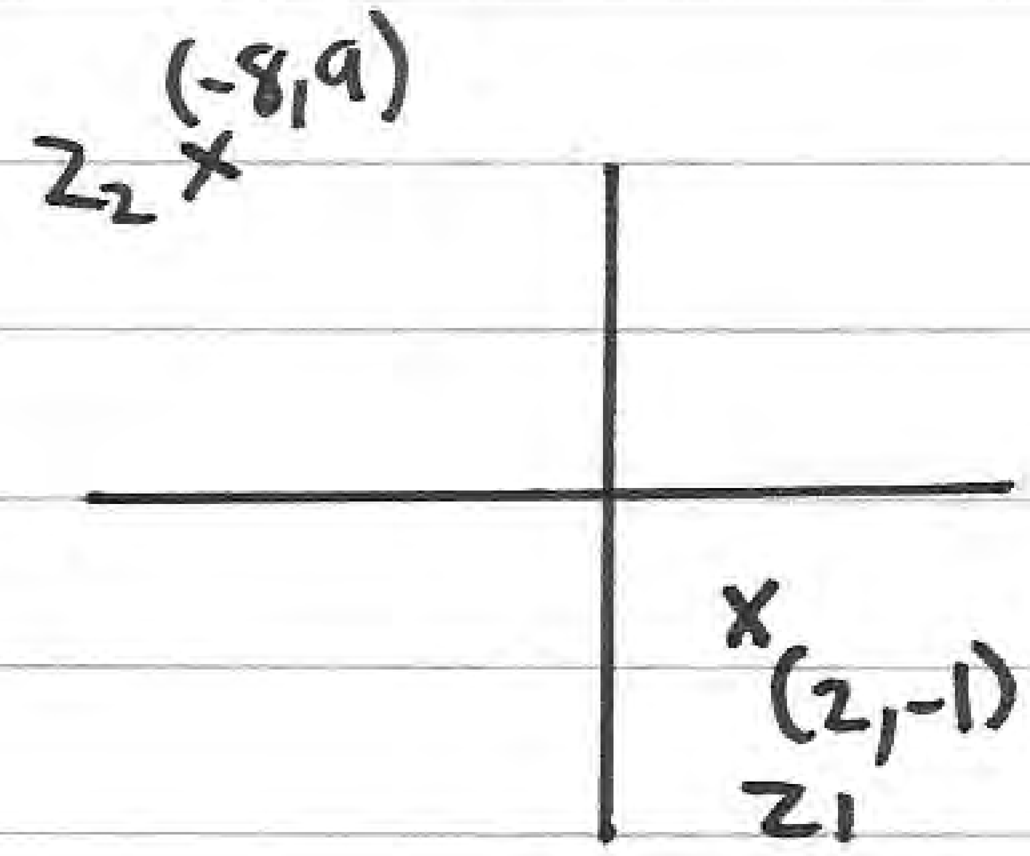
FP1 June 2009

1) $z_1 = 2 - i$
$z_2 = -8 + 9i$



b) $|z_1| = \sqrt{2^2 + 1^2} = \sqrt{5}$

c) $\arg(z_1) = -\tan^{-1}\left(\frac{1}{2}\right) = \underline{-0.46^c}$

d) $\frac{z_2}{z_1} = \frac{-8+9i}{2-i} \times \frac{2+i}{2+i} = \frac{-16-8i+18i+9i^2}{4-i^2} = \frac{-25+10i}{5}$

$= \underline{-5+2i}$

2) $\sum r(r+1)(r+3) = \sum r^3 + 4r^2 + 3r = \sum r^3 + 4\sum r^2 + 3\Sigma$

$= \frac{1}{4}n^2(n+1)^2 + 4\left(\frac{1}{6}n(n+1)(2n+1)\right) + 3\left(\frac{1}{2}n(n+1)\right)$

$= \frac{3}{12}n^2(n+1)^2 + \frac{8}{12}n(n+1)(2n+1) + \frac{18}{12}n(n+1)$

$= \frac{1}{12}n(n+1)\left(3n(n+1) + 8(2n+1) + 18\right)$

$= \frac{1}{12}n(n+1)\left(3n^2 + 19n + 26\right)$

$= \frac{1}{12}n(n+1)(n+2)(3n+13) \quad \Rightarrow \underline{k = 13}$

b) $\sum_{21}^{40} r(r+1)(r+3) = \frac{1}{12}(40)(41)(42)(133) - \frac{1}{2}(20)(21)(22)(73)$

$= 707210$

3) $f(x) = (x^2+4)(x^2+8x+25)$

$x^2 = -4 \Rightarrow x = \pm 2i$

$x^2+8x = -25 \Rightarrow (x+4)^2 - 16 = -25 \Rightarrow (x+4)^2 = -9$

$\Rightarrow x+4 = \pm 3i \Rightarrow x = -4 \pm 3i$

$\alpha = 2i, -2i, -4+3i, -4-3i$

b) $2i - 2i - 4 + 3i - 4 - 3i = \underline{-8}$

4) $f(x) = x^3 - x^2 - 6$ $\qquad f(2.2) = -0.192 \quad < 0$

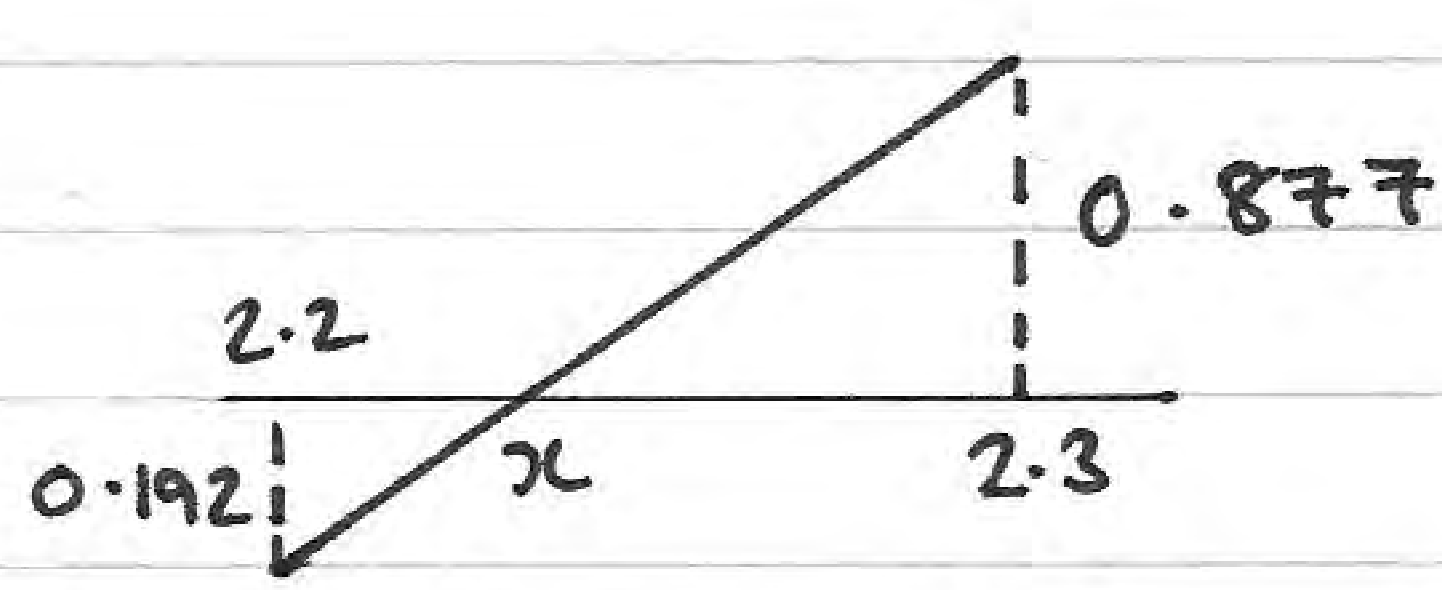
$\qquad f(2.3) = 0.877 \quad > 0$

Sign change $\Rightarrow$ $\alpha \in [2.2, 2.3]$

b) $f'(x) = 3x^2 - 2x$

$$x_1 = 2.2 - \frac{f(2.2)}{f'(2.2)} = \underline{2.219}$$

c)



$$\frac{x-2.2}{0.192} = \frac{2.3-x}{0.877}$$

$0.877x - 1.9294 = 0.4416 - 0.192x$

$1.069x = 2.371$

$x = \underline{2.218}$

5) $R = \begin{pmatrix} a & 2 \\ a & b \end{pmatrix}$ $\quad R^2 = \begin{pmatrix} a & 2 \\ a & b \end{pmatrix}\begin{pmatrix} a & 2 \\ a & b \end{pmatrix} = \begin{pmatrix} a^2+2a & 2a+2b \\ a^2+ab & 2a+b^2 \end{pmatrix}$

$R^2 = \begin{pmatrix} 15 & 0 \\ 0 & 15 \end{pmatrix}$ $\quad \Rightarrow a^2+2a = 15 \Rightarrow a^2+2a-15=0$

$(a+5)(a-3)=0$

$\Rightarrow \underline{a=3}$ Since $a>$

$2a+2b=0 \Rightarrow \underline{b=-3}$

6) $y^2 = 16x = 4ax \Rightarrow a = 4$ focus $S(4,0)$

directrix $x+4=0$

a) $x = 4t^2 \Rightarrow y^2 = (4t^2)\times 16 = 64t^2$

$y = \sqrt{64t^2} = 8t$ #

b) $S(4,0)$

c) $y^2 = 16x \Rightarrow \frac{d}{dx}y^2 = \frac{d}{dx}16x \Rightarrow 2y\frac{dy}{dx} = 16$

$\Rightarrow \frac{dy}{dx} = \frac{8}{y} = M_t \Rightarrow M_n = \frac{-y}{8} = \frac{-8t}{8} = -t$

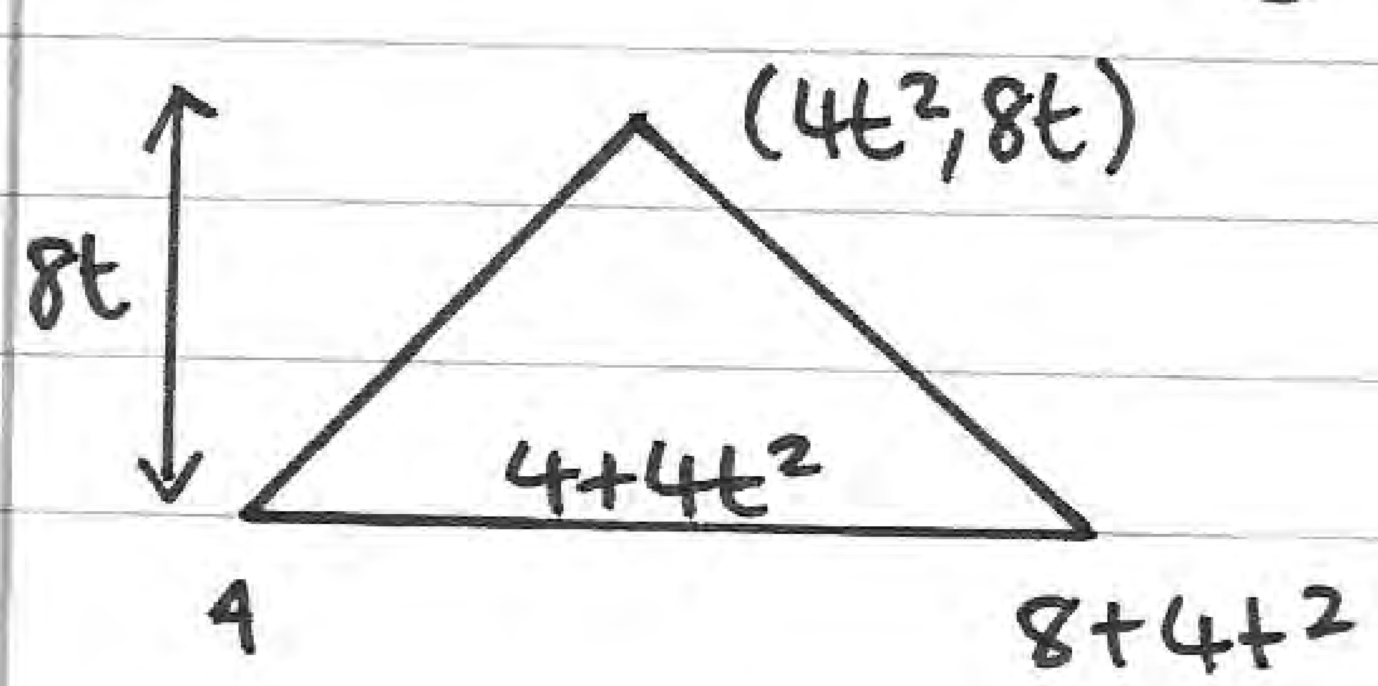
alt $y^2 = 16x \Rightarrow y = 4x^{\frac{1}{2}}$ $\quad \frac{dy}{dx} = 2x^{-\frac{1}{2}} = \frac{2}{\sqrt{4t^2}} = \frac{2}{2t}$

$y - 8t = -t(x-4t^2) \Rightarrow y - 8t = -tx + 4t^3$

$\Rightarrow y + tx = 8t + 4t^3$ #

c) meets x when $y=0 \Rightarrow tx = 8t+4t^3$

$\Rightarrow x = 8+4t^2$



$\text{Area} = \frac{1}{2}(4+4t^2)\times 8t$

$= \underline{16t + 16t^3}$

7) $A = \begin{pmatrix} a & -2 \\ -1 & 4 \end{pmatrix}$ Singular $\Rightarrow \det A = 0 \Rightarrow 4a - 2 = 0$

$\Rightarrow a = \frac{1}{2}$

b) $B = \begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}$ $\det B = 12 - 2 = 10$ $B^{-1} = \frac{1}{10}\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$

c) $BP = Q$ $B^{-1}BP = B^{-1}Q \Rightarrow P = B^{-1}Q$

$$P = \frac{1}{10}\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} k-6 \\ 3k+12 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 4k - 24 + 6k + 24 \\ k - 6 + 9k + 36 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 10k \\ 10k+30 \end{pmatrix}$$

$P = \begin{pmatrix} k \\ k+3 \end{pmatrix}$ $P(k, k+3)$ $y = x + 3$

if $x = k$ $y = k + 3$ as required.

8) $n=1$ $f(1) = 5^1 + 8(1) + 3 = 16 = 4 \times 4 \Rightarrow$ divisible by 4

$n=k$ $f(k) = 5^k + 8k + 3$ assume is divisible by 4

$n = k+1$ $f(k+1) = 5^{k+1} + 8(k+1) + 3$

$\Rightarrow f(k+1) = 5(5^k) + 8k + 11$

$$f(k+1) - f(k) = (5(5^k) + 8k + 11) - (5^k + 8k + 3)$$
$$= 4(5^k) + 8$$
$$= 4[(5^k) + 2]$$

$\Rightarrow f(k+1) = 4[(5^k) + 2] + f(k)$

true for $n=1$

true for $n=k+1$ if true $n=k$

$\therefore$ by induction true for all $n \in \mathbb{Z}^+$

b) $n=1$ $\quad \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^1 = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$

$$\begin{pmatrix} 2n+1 & -2n \\ 2n & 1-2n \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$$ true for $n=$

$n=k$ $\quad \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^k = \begin{pmatrix} 2k+1 & -2k \\ 2k & 1-2k \end{pmatrix}$ assume true

$n=k+1$ $\quad \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^{k+1} = \begin{pmatrix} 2k+3 & -2k-2 \\ 2k+2 & -1-2k \end{pmatrix}$

$$\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^k$$

$$= \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2k+1 & -2k \\ 2k & 1-2k \end{pmatrix}$$

$$= \begin{pmatrix} 6k+3-4k & -6k-2+4k \\ 4k+2-2k & -4k-1+2k \end{pmatrix}$$

$$= \begin{pmatrix} 2k+3 & -2k-2 \\ 2k+2 & -1-2k \end{pmatrix}$$ as required.

true for $n=1$, true for $n=k+1$ if true for $n=k$

$\therefore$ by Induction true for all $n \in \mathbb{Z}^+$